Multiplicity of Periodic Solutions for Differential Equations Arising in the Study of a Nerve Fiber Model

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Abstract

We deal with the periodic boundary value problem for a second-order nonlinear ODE which includes the case of the Nagumo type equation $v_{xx} - gv + n(x)F(v) = 0$, previously considered by Grindrod and Sleeman and by Chen and Bell in the study of the model of a nerve fiber with excitable spines. In a recent work we proved a result of nonexistence of nontrivial solutions as well as a result of existence of two positive solutions, the different situations depending by a threshold parameter related to the integral of the weight function $n(x)$. Here we show that the number of positive periodic solutions may be very large for some special choices of a (large) weight n. We also obtain the existence of subharmonic solutions of any order. The proofs are based on the Poincaré - Bikhoff fixed point theorem.

Key words: Nagumo type equation, periodic solutions, subharmonics, rotation numbers, Poincaré - Birkhoff fixed point theorem.
2000 AMS subject classification : 34C25, 37E40, 92C20.

1 Introduction

This paper deals with the study of periodic solutions to a class of second order ordinary differential equations arising in the search of stationary solutions of a partial differential system modelling a nerve fiber with excitable spines (see [3], [10]).

The model equation considered in [3, pp.391–395] takes the form

$$v_{xx} - gv + n(x)F(v) = 0 \tag{1}$$

where $g > 0$ is a given constant, $n(\cdot)$ is a positive β-periodic piecewise constant function and $F : \mathbb{R} \to \mathbb{R}$ is a smooth mapping with $-F$ having a N-shape. Typically, $F(s)$ has three zeros $0 < a < 1$ and $F(s) > 0$ for $s \in (-\infty, 0[\cup]a, 1[$, $F(s) < 0$ for $s \in]0, a[\cup]1, +\infty)$. According to [3, pp.391–392], a possible example for F is given by $F(s) = f(\tilde{R}(s))$, where $f(u) = u(1-u)(u-a)$ and $\tilde{R}$ is a smooth monotone increasing function with $\tilde{R}(0) = 0$, $\tilde{R}(a) = a$ and $\tilde{R}(1) = 1$. Actually, equation (1) comes from the system

$$\begin{cases} v_{xx} - gv + n(x)f(v) = 0 \\ v = S(u;\rho) := u - \rho^{-1}f(u) \end{cases}$$

by assuming the function $S(u)$ invertible with inverse $\tilde{R}(v)$. Such an assumption is satisfied when the spine stem resistance $R_S = \rho^{-1}$ approaches zero (cf. [3, pp.386–387]).

Equations of the form of (1) present a typical threshold phenomenon that may be easily described if we think for a moment at the weight $n(x)$ as a constant function and look for constant solutions. It is clear that if $n = n_0$ is a small positive number, then the only β-periodic solution is the trivial one. On the other hand, if $n = n_1$ is sufficiently large, the line $y = gs$ intersects the $\backslash\!\wedge\!\backslash$-shaped curve $y = n_1 F(s)$ in two nontrivial points (at least) and therefore we have the existence of at least two positive β-periodic solutions (see Figure 1).

In [3] the authors, using a phase-plane analysis and gluing together the solutions found for different values of the coefficients, showed that a similar result can be obtained for a piecewise constant weight $n(x)$ which, in a period, takes two values $n_1 > n_0 > 0$ (with $n(x) = n_0$ on $]\alpha, \beta[$ and $n(x) = n_1$ on $]0, \alpha[$). Then, they obtained [3, Lemma 4.1] the existence of the only trivial solution if α and n_0 are small (which means that the weight is close to a small value n_0 for the main part of the time) and the existence of one positive β-periodic solution if n_1 is sufficiently large and α is not too close to zero (which means that the weight is large for an adequate amount of time). In a further result of the same paper, the authors [3, Lemma 4.3] also claimed the existence of "many" periodic solutions when $S(u)$ is not invertible and n_1 is large enough. The proof, however, in this case is only suggested "by reading the superimposed phase portraits". For all these results, some further technical conditions (that we do not recall here) were assumed. Among such conditions, an hypothesis which implies

$$\int_0^1 F(s)\,ds > 0, \tag{2}$$

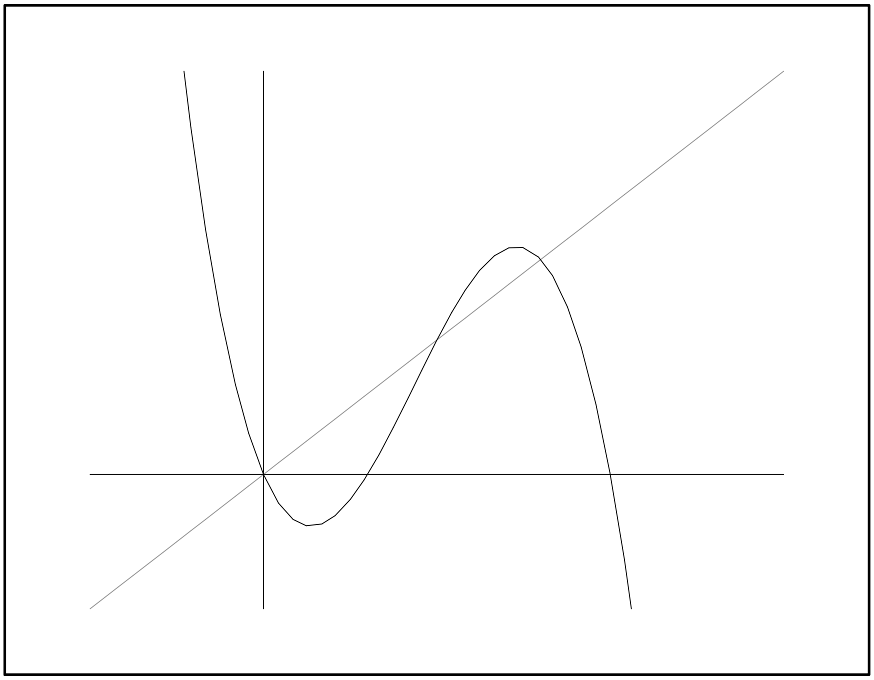

Fig. 1. *Intersections of $y = nF(s)$ with $y = gs$.*

was required.

A few questions may arise from these conclusions in [3], in particular, concerning the possibility of extending Chen and Bell results to a general class of weight functions and to the existence of at least two positive β-periodic solutions when the weight is sufficiently large (in some sense), as well as to prove in a more general way (i.e., using some argument which is not based by reading the superimposed phase portraits and therefore applicable only to weights which are two-step functions or small perturbations of two-steps functions [3, Remark 4.6]) the existence of "many" periodic solutions for large but arbitrarily shaped profiles.

In a previous recent article [16], we addressed our attention to the first two questions, as well as to some related problems. In particular, we showed that the same features are preserved for a broad class of functions F and for an arbitrary positive β-periodic weight $n \in L^1([0,\beta])$. In this case, the L^1-norm (or the mean value) of $n(x)$ plays the role of a threshold parameter in the sense that we have only the trivial solution when $|n|_1$ is small and at least two positive β-periodic solutions when $|n|_1$ is sufficiently large. Indeed, these results are proved in [16] for rather general second order equations including (1) as a particular case. The mathematical tools employed in [16] were based on upper and lower solutions, the study of quadratic forms in Hilbert spaces and critical point theory. Also in [16] condition (2) was needed for the proof of the existence of two nontrivial solutions.

It is the aim of this work to consider now the third question, that is, to prove the existence of a large number of periodic solutions (both harmonics and subharmonics), for a general (i.e., not necessarily piecewise constant) positive periodic weight function. Note that now the elementary observation made above which suggested to look at the intersections of the graph of $y = nF(s)$ with the line $y = gs$ in order to prove the existence of at least two periodic solutions also for a non-constant weight does not work anymore. Here we have better to look at the phase-plane portrait of the planar system

$$\begin{cases} v' = y \\ y' = g\,v - n(x)F(v) \end{cases} \tag{3}$$

and observe that if $n(x) = n$ is a constant function, with n sufficiently large and $F'(a) > 0$ then, near the point $(a, 0)$ an equilibrium point $(a_n, 0)$ appears and such an equilibrium point is a local center surrounded by a family of periodic orbits contained in the strip $]0, 1[\,\times\mathbb{R}$. The fundamental period of these orbits becomes larger as their energy increases, a situation which is reminiscent to the one encountered in the study of the trajectories approaching the separatrices in the nonlinear simple pendulum equation (see Figure 2).

This point of view addresses our investigation toward the search of periodic trajectories "near" $(a, 0)$ also when $n(x)$ is not constant, provided that $n(x)$ has a large principal part. Indeed, we will show that such periodic trajectories actually exist (see Theorem 3.1 of Section 3). More precisely, using the Poincaré - Birkhoff fixed point theorem we shall prove that, if we split

$$n(x) = \bar{n} + \tilde{n}(x), \tag{4}$$

where $\bar{n}$ a suitably chosen constant value (for instance, in some situations, we could take $\bar{n} = \dfrac{1}{\beta}\displaystyle\int_0^\beta n(x)\,dx$, but other possible choices may be suitable as well, depending on the weight function) then, the number of positive β-periodic solutions becomes large if $\bar{n}$ grows to infinity and $|\tilde{n}|_1$ is small (in a suitable sense). Moreover, the same fact is true also with respect to the subharmonic solutions (that is the $m\beta$-periodic solutions having $m\beta$ as their minimal period). To this aim, we shall consider Eq. (1) as a modification of the autonomous equation

$$v_{xx} - gv + \bar{n}F(v) = 0$$

and treat the remaining term $\tilde{n}(x)F(v)$ as a perturbation (see, for instance, [2], [8], [9], [11] for a similar approach in the study of some different equations).

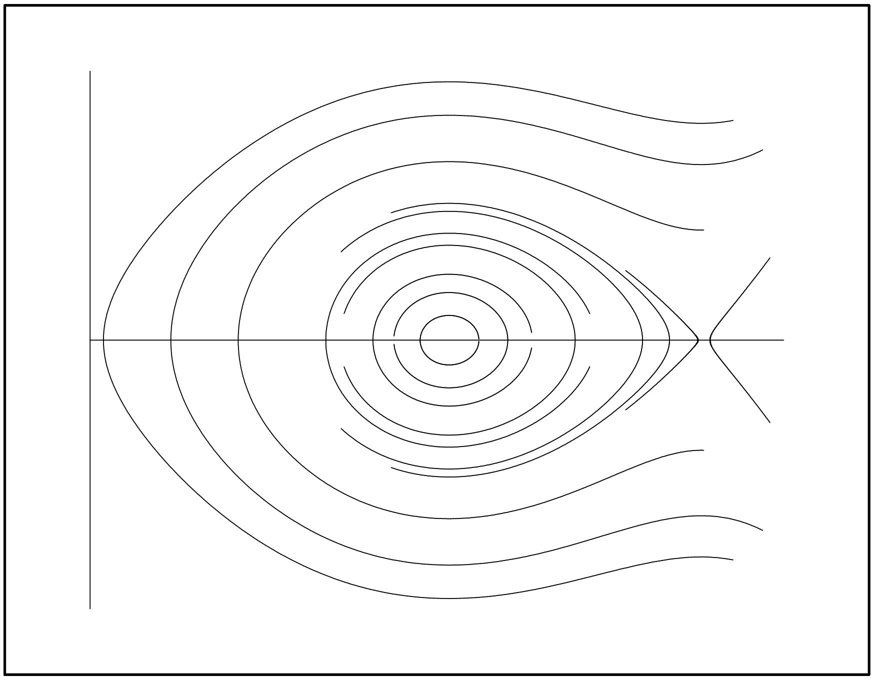

Fig. 2. *Phase-portrait of system* (3) *for* $n(x) = n = 20$ *(constant),* $g = 0.1$ *and* $F(x) = x(1-x)(x-a)$, *with* $a = 0.6$. *Plotting the orbit-segments at different initial points* $(x_0, 0)$ *with* $x_0 \in]0,1]$, *but along a same time interval* $[-T, T]$, *one can see the presence of periodic trajectories around the stable equilibrium point* $(a_n, 0)$ *which is near to* $(a, 0)$. *The period needed to complete one turn becomes larger as the trajectories approach the separatrix.*

As a final remark, we observe that the only crucial assumption for our main multiplicity result of Section 3 is the existence of a point $a \in]0,1[$ where

$$F(a) = 0, \quad F'(a) > 0,$$

an hypothesis which is always satisfied for the typical N-functions considered in the literature. On the other hand, with our approach, we do not need any condition like (2) on $\int_0^1 F(s)\,ds$. As pointed out before, we recall that the positivity of the integral in (2) was required both in the phase-plane analysis approach of Chen and Bell [3] (based on the superposition of the phase portraits for $n = n_0$ and $n = n_1$) as well as in the variational approach of our recent work (based on Ambrosetti - Rabinowitz mountain pass theorem [1]).

Hopefully, the study of equation (1) or its variants may be relevant for the attempt of better understanding the transmission of the impulses along the nerve fibers also with respect to the fact that there are some serious pathologies which may arise as a consequence of a deficient distribution of myelin along the nerve axon (disorders of myelination). The weight function $n(x)$ in the Nagumo equation represents the axial distribution of the myelin along the axoplasm. It is known that some "fat" areas alternate with some gaps (the so-called nodes of Ranvier). Such a profile for the distribution of $n(x)$ has lead

some authors (like in [3], [10]) to the study of a piecewise constant weight which alternates between a small and a large value. However, the real images of a myelinated axon show that the true shape of the profile sometimes may be quite far from this idealized picture and therefore results like those in [3] could not be applied. Our theorems show that those conclusions are still valid for general positive weights and \/\-shaped functions.

2 Preliminary results and notation

We recall in this section an auxiliary result (taken from [16]), about the periodic solutions of (1), which is useful for what follows.

The nerve fiber equation (1), has the form of a second-order equation

$$v'' + h(x, v) = 0 \tag{5}$$

where $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is β-periodic in its first variable, that is, $h(\cdot, s)$ is measurable for all $s \in \mathbb{R}$ and there is $\beta > 0$ such that $h(x+\beta, s) = h(x, s)$ for almost every $x \in \mathbb{R}$ and all $s \in \mathbb{R}$, $h(x, \cdot)$ is continuous for almost every $x \in [0, \beta]$ and, for every $r > 0$ there is a measurable function $\rho_r \in L^1([0, \beta], \mathbb{R}^+)$ such that $|h(x, s)| \leq \rho_r(x)$ for almost every $x \in [0, \beta]$ and every $s \in [-r, r]$. Solutions of (5) are considered in the Carathéodory sense too.

We look for the existence of $m\beta$-periodic solutions to Eq. (5), for some positive integer m, that is for solutions of problem

$$(P_m) \qquad v'' + h(x, v) = 0, \quad v(x + m\beta) = v(x),\ \forall\, x \in \mathbb{R},$$

or, equivalently, $v'' + h(x, v) = 0$, with $v(m\beta) - v(0) = v'(m\beta) - v'(0) = 0$.

Clearly, in the model of our interest, we have

$$h(t, s) := -gs + n(t)F(s) \tag{6}$$

and the N-shape of $-F(s)$ implies that for $h(t, s)$ the following conditions are satisfied.

$$(A_1) \qquad h(x, 0) \equiv 0$$

and, for a.e. $x \in [0, \beta]$,

$$(A_2) \qquad h(x, s) > 0,\ \forall\, s < 0, \qquad h(x, s) < 0,\ \forall\, s \geq 1.$$

Under these assumptions, we obtain:

Lemma 2.1 *[16] Suppose (A_1) and (A_2) hold. Then, any possible solution $v(\cdot)$ of problem (P_m) satisfies $0 \leq v(x) \leq 1, \forall x \in \mathbb{R}$. Moreover, if h is (locally) lipschitzean at $s = 0$, then any nontrivial solution of (P_m) is strictly positive and, if h is (locally) lipschitzean at $s = 1$, then any solution of (P_m) is strictly less than one.*

The proof is based on some direct estimates and classical arguments from the theory of upper and lower solutions (see, e.g., [4], [5]). We refer to [16] for more details and remarks.

Thanks to Lemma 2.1 we can confine ourselves to the solutions $v(x)$ belonging to the interval $[0, 1]$ and, moreover, we can modify as we like the function $h(t, s)$ (respectively the function $F(s)$ in (1)) on $s \in \mathbb{R} \setminus [0, 1]$ and such a modification will have no effect on the existence of the periodic solutions as long as the sign conditions outside the interval $[0, 1]$ are preserved.

In the sequel, standard notation is used. We only warn that for a β-periodic locally integrable function $u(\cdot)$, we denote by $|u|_1$ its L^1-norm on a interval of length β. Sometimes (when no confusion may occur) the same symbol will be used to denote the L^1-norm on $[0, m\beta]$, when we are interested in the search of $m\beta$-periodic functions.

3 Multiplicity results

Now we focus our attention on equation (1) and assume that $g > 0$ is a fixed constant and $F : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function such that $F(0) = F(a) = F(1) = 0$ with $a : \ 0 < a < 1$ and such that $F(s) > 0$ for $s \in (-\infty, 0[\cup]a, 1[$, $F(s) < 0$ for $s \in]0, a[\cup]1, +\infty)$. We also suppose that $n(\cdot) : \mathbb{R} \to \mathbb{R}^+$ is a β-periodic locally integrable function.

We are going to show an application of the Poincaré - Birkhoff fixed point theorem to the search of the periodic solutions of (1). To this aim and for technical reasons it will be convenient to modify suitably the function F outside the interval $[0, 1]$. Precisely, let us set

$$\delta(s) := \max\{0, \min\{s, 1\}\}$$

and define

$$F_0(s) := F(\delta(s)) + k_0\ell(s),$$

where $\ell(s)$ is given by

$$\ell(s) = \begin{cases} \exp(1/s), & s < 0 \\ 0, & 0 \leq s \leq 1 \\ -\exp(1/(1-s)), & s > 1, \end{cases} \tag{7}$$

and

$$0 < k_0 \leq c_0 := \max_{s\in[0,1]} |F(s)|.$$

With these positions, we have that $F_0 : \mathbb{R} \to \mathbb{R}$ is a bounded (globally) Lipschitz continuous function which is smooth in $]0,1[$, $F_0(s) = F(s)$ for all $s \in [0,1]$, $F_0(s) > 0$ for $s < 0$ and $F_0(s) < 0$ for $s > 1$. Moreover,

$$\sup_{s\in\mathbb{R}} |F_0(s)| = \max_{s\in\mathbb{R}} |F_0(s)| = \max_{s\in[0,1]} |F(s)| = c_0\,.$$

A straightforward application of Lemma 2.1 ensures that for each positive integer m, all the possible nontrivial $m\beta$-periodic solutions of equation

$$v'' - gv + n(x)F_0(v) = 0 \tag{8}$$

have range into the open interval $]0,1[$ and so they are, indeed, solutions of equation (1).

The fact that after our modification the nonlinearity is now bounded, suggests the possibility of splitting $n(x)$ as in (4) and writing equation (8) as

$$v'' - gv + \bar{n}F_0(v) = p(x,v) \tag{9}$$

which looks like a perturbation of the autonomous equation

$$v'' - gv + \bar{n}F_0(v) = 0 \tag{10}$$

by an external source

$$p(x,v) := -\tilde{n}(x)F_0(v).$$

Now we are in position to prove our main multiplicity result.

Theorem 3.1 *Assume that*

$$F'(a) > 0.$$

Then, for every integer $N \geq 1$, there exists a (large) value $\mu = \mu_N > 0$ such that for each $\mu_2 \geq \mu_1 > \mu$ there is a (small) value $\varepsilon = \varepsilon_{N,\mu_1,\mu_2} > 0$ such that for each $n(\cdot)$ with

$$\bar{n} \in [\mu_1, \mu_2] \quad \textit{and} \quad |\tilde{n}|_1 < \varepsilon$$

there are at least $2N$ solutions of equation (1) which are β-periodic and take values in $]0,1[$.

The proof of Theorem 3.1 is based on the Poincaré - Birkhoff fixed point theorem and will be split into some steps.

First of all, we write equation (8) as a first order planar system of the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -h(t,x) \end{cases} \tag{11}$$

with

$$h(t,s) = -gs + n(t)F_0(s), \quad \dot{x} = \frac{dx}{dt}.$$

Note that we have changed the name to the variables, by denoting the x and the v variables in (8) as t and x, respectively. According to our hypotheses, $h : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is β-periodic in t and satisfies a global Lipschitz condition

$$|h(t,s_1) - h(t,s_2)| \le A(t)|s_1 - s_2|, \quad \text{for all } s_1\,, s_2\, \in \mathbb{R} \text{ and for a.e. } t \in \mathbb{R},$$

where $A(\cdot)$ is a suitable β-periodic measurable function with $A(\cdot) \in L^1([0,\beta])$. In our case, we actually have $A(t) = g+n(t)L_0$ where L_0 is a Lipschitz constant for F_0 (which is globally lipschitzean on $\mathbb{R}$). As a consequence, we have that for each $z_0 = (x_0,y_0) \in \mathbb{R}^2$ and $t_0 \in \mathbb{R}$, system (11) has a unique solution $\zeta(t) = \zeta(t,t_0,z_0) = (x(t,t_0,z_0), y(t,t_0,z_0))$ satisfying the initial condition $\zeta(t_0) = z_0$, with $\zeta(t)$ defined for all $t \in \mathbb{R}$. Hence, the Poincaré's map

$$\phi : z_0 \mapsto \zeta(\beta, 0, z_0)$$

is well defined as a homeomorphism of $\mathbb{R}^2$ onto itself.

Next, we introduce a rotation number associated to ϕ with respect to a given point $q_0 \in \mathbb{R}^2$.

Let $m \in \mathbb{N}$ be a positive integer and let $q_0 = (q_1^0, q_2^0)$ and $z_0 = (x_0, y_0)$ be such that

$$\zeta(t,0,z_0) \neq q_0 \quad \forall\, t \in [0, m\beta].$$

We define the rotation number (see, for instance, [7], [15]) as

$$\mathrm{rot}_m(z_0,q_0) = \frac{1}{2\pi}\int_0^{m\beta} \frac{y(t)^2 + x(t)h(t,x(t))}{(x(t)-q_1^0)^2 + (y(t)-q_2^0)^2}\, dt$$

which represents the normalized angular displacement around the point q_0 of the solution $\zeta(t,0,z_0)$ for t varying along the time interval $[0,m\beta]$. In fact, if we use the Prüfer transformation and write the solution $\zeta(t)$ in polar coordinates (θ,ρ) with respect to the point q_0 , we have that $\rho(t)^2 = ||\zeta(t,0,z_0) - q_0||^2 = (x(t,0,z_0) - q_1^0)^2 + (y(t,0,z_0) - q_2^0)^2 > 0$ for all $t \in [0,m\beta]$ and therefore the

number $\theta(t) - \theta(0)$ is well defined. It turns out that

$$\mathrm{rot}_m(z_0, q_0) = \frac{\theta(0) - \theta(m\beta)}{2\pi}$$

and this number counts as positive the clockwise rotations around the point q_0 .

Our main tool to prove the existence of periodic solutions is the following result which is adapted to our situation from W. Ding's generalized version of the Poincaré - Birkhoff theorem [8].

Theorem 3.2 *Let $q_0 = (q_1^0, q_2^0) \in \mathbb{R}^2$ and let D_0 be an open neighborhood of q_0 such that*

$$\zeta(t, t_0, z_0) \neq q_0\,, \quad \forall\, z_0 \in \partial D_0\,, \ \ \forall\, t_0 \in [0, m\beta[\,, \ \forall\, t \in [t_0, m\beta]. \qquad (12)$$

Suppose that $\Gamma \subset \mathbb{R}^2 \setminus D_0$ is a simple closed curve which is star-shaped with respect to q_0 and there exist a positive integer j such that

$$\mathrm{rot}_m(z_1, q_0) > j, \quad \forall\, z_1 \in \Gamma. \qquad (13)$$

Furthermore, let us assume there is a (sufficiently large) radius $R > 0$, with $\Gamma \subseteq B(q_0, R)$ such that

$$\mathrm{rot}_m(z_2, q_0) < 1, \quad \forall\, ||z_2|| = R. \qquad (14)$$

Define $\mathcal{A}$ to be the open annulus bounded by Γ and $\partial B(q_0, R)$. Then, there are at least $2j$ solutions $\tilde{\zeta}_k$ and $\hat{\zeta}_k$ (for $k = 1, \ldots, j$) of (11) which are $m\beta$-periodic and such that $\tilde{\zeta}_k(0), \hat{\zeta}_k(0) \in \mathcal{A}$ and

$$\mathrm{rot}_m(z_0, q_0) = k, \quad \textit{for} \ \ z_0 = \tilde{\zeta}_k(0) \ \textit{or} \ \ z_0 = \hat{\zeta}_k(0).$$

In the particular case when $q_0 = (q_1^0, 0)$ we have also that, setting $\tilde{\zeta}_k(t) = (\tilde{u}_k(t), \tilde{y}_k(t))$ and $\hat{\zeta}_k(t) = (\hat{u}_k(t), \hat{y}_k(t))$, it follows that $\tilde{u}_k$ and $\hat{u}_k$ are $m\beta$-periodic solutions of equation

$$u'' + h(t, u) = 0$$

with $\tilde{u}_k(t) - q_1^0$ and $\hat{u}_k(t) - q_1^0$ having precisely $2k$ simple zeros in the interval $[0, m\beta[$.

Proof. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the Poincaré's map associated to system (11). By the Liouville theorem it follows that ϕ is area-preserving and therefore ϕ^m is an area-preserving planar homeomorphism as well. Let E_0 be the open bounded set confined by the curve Γ. By the assumption, $q_0 \in D_0 \subset E_0$ and $\partial E_0 = \Gamma$. Note also that the rotation number $\mathrm{rot}_m(z_0, q_0)$ is well defined for each $z_0 \in \mathbb{R}^2 \setminus E_0$ due to the fact $\zeta(t, 0, z_0) \neq q_0$ for all $t \in [0, m\beta]$

and each $z_0 \notin E_0$. Indeed, if, by contradiction, there is $z_0 \notin E_0$ such that $\zeta(t^*) = \zeta(t^*, 0, z_0) = q_0$ for some $t^* \in]0, m\beta]$, then $\zeta(t_1, 0, z_0) = z_1 \in \partial D_0$ for some $t_1 \in [0, t^*[$. This means that $\zeta(t^*) = \zeta(t^*, t_1, z_1) = q_0$ and we have a contradiction to hypothesis (12). Now, if we restrict $\psi := \phi^m$ to the closed disk $B[q_0, R]$ we have that

$$\psi^{-1}(q_0) \in E_0$$

and ψ satisfies the twist condition with respect to the inner and the outer boundaries of the closed annulus $\bar{\mathcal{A}}$ for each integer $k \in [1, j]$. Namely, we have $\mathrm{rot}_m(\cdot, q_0) > k$, on Γ and $\mathrm{rot}_m(\cdot, q_0) < k$, on $\partial B(q_0, R)$. The W. Ding's generalized version of the twist theorem guarantees the existence of at least two fixed points in $\mathcal{A}$ for ψ. These fixed points are initial values (at the time $t = 0$) of two $m\beta$-periodic solutions $\tilde{\zeta}_k(\cdot)$ and $\hat{\zeta}_k(\cdot)$ of system (11), respectively. Moreover, the rotation number associated to $\tilde{\zeta}_k$ and $\hat{\zeta}_k$ is equal to k.

In the special case when $q_0 = (q_1^0, 0)$, and by virtue of the particular form of system (11), we know that the result about the rotation numbers implies that the first coordinates $\tilde{u}_k$ and $\hat{u}_k$ of $\tilde{\zeta}_k$ and $\hat{\zeta}_k$ crosses the line $u = q_1^0$ exactly $2k$ times in the interval $[0, m\beta[$. □

Remark 3.1 Clearly, if $u(\cdot)$ is a $m\beta$-periodic solution of the equation $u'' + h(t, u) = 0$ with $h(t + \beta, u) = h(t, u)$ then

$$u(\cdot + \beta), \dots, u(\cdot + j\beta), \dots, u(\cdot + (m-1)\beta)$$

are $m\beta$-periodic solutions as well. We consider these solutions as equivalent each other and we say that they belong to the same periodicity class. A further consequence of the Poincaré - Birkhoff theorem (as pointed out in [14]) ensures that the solutions $\tilde{u}_k$ and $\hat{u}_k$ (for which the existence is claimed in Theorem 3.2 do not belong to the same periodicity class. Obviously, also the solutions $\tilde{u}_k$, $\hat{u}_k$ $\tilde{u}_\ell$, $\hat{u}_\ell$ for $\ell \neq k$ belong to different periodicity classes (in fact their rotation numbers are different). We refer also to [13], [18] for a throughout discussion concerning this aspect.
Moreover, we observe that the information about the associated rotation numbers permits obtain some conclusions about the minimality of the period. For instance, if $m \geq 2$ and $k \geq 1$ are co-prime numbers (that is m/k is not further reducible), then it is possible to prove that the $m\beta$-periodic solutions $\tilde{u}_k$ and $\hat{u}_k$ are not $i\beta$-periodic for each $i = 1, \dots, m-1$. In particular, for $k = 1$, we have that the solutions we find have $m\beta$ as their minimal period (see [7] where this discussion is carried on with more details).

In the proof of Theorem 3.1 using Theorem 3.2 we take

$$q_0 = (a_{\bar{n}}, 0)$$

with $0 < a_{\bar{n}} < 1$ such that

$$\frac{F_0(a_{\bar{n}})}{a_{\bar{n}}} = \frac{g}{\bar{n}}$$

Note that

$$a_{\bar{n}} > a, \quad \forall\, \bar{n} > 0 \quad \text{and} \quad \lim_{\bar{n}\to+\infty} a_{\bar{n}} = a.$$

Our next result shows that large solutions rotate slowly.

Lemma 3.1 *There is $R_0 = R_0(m, |n|_1) > 0$ such that for each initial point z_2 with $||z_2|| \geq R_0$*

$$\mathrm{rot}_m(z_2, q_0) < 1, \quad \forall\, ||q_0|| \leq 1.$$

Proof. First of all, we recall a well known consequence of the global existence of the solutions (cf. [12]), that is, *there is a continuous nondecreasing function $\eta : \mathbb{R}^+ \to \mathbb{R}_0^+$, with $\eta(r) > r$ for all $r \geq 0$, such that*

$$||\zeta(t, t_0, z_0)|| > R, \quad \forall\, t, t_0 \in [0, m\beta],\ \forall\, ||z_0|| \geq \eta(R).$$

The function η depends upon the constants which bound the growth of $h(t, x)$ and hence, we have $\eta = \eta_{m,|n|_1}$ (see also [17, Lemma 2]).

Let $R > 0$ be such that $||\zeta(t)|| \geq R$ for all $t \in [0, m\beta]$. We claim that if $R > 1$ then

$$\mathrm{rot}_m(\zeta(0), 0) < 1/2.$$

Indeed, let us assume the contrary and suppose that $\mathrm{rot}_m(\zeta(0), 0) \geq 1/2$. This implies that the projection $\zeta(t)/||\zeta(t)||$ of $\zeta(t)$ to S^1 covers at least one of the intersections of S^1 with the standard quadrants in the $x\,y$-plane. Just to fix the ideas, let us assume that the trajectory crosses the first quadrant. Hence, there are $0 \leq t_1 < t_2 \leq m\beta$ such that

$$\begin{array}{l} \zeta(t_1) = (x(t_1), y(t_1)) = (1, y_1), \quad \text{with } \ y_1 \geq (R^2 - 1)^{\frac{1}{2}} \\ \zeta(t_2) = (x(t_2), y(t_2)) = (x_2, 0), \quad \text{with } \ x_2 \geq R \\ ||\zeta(t)|| \geq R, \quad x(t) > 1, \quad y(t) > 0, \quad \forall\, t \in \left]t_1, t_2\right[\,. \end{array}$$

Then, passing to the polar coordinates (via the Prüfer transformation), we find, for $t \in [t_1, t_2]$ and using the fact that $h(t, s)s \leq 0$, for all $s \geq 1$,

$$-\theta'(t) = \frac{y(t)^2 + x(t)h(t, x(t))}{x(t)^2 + y(t)^2} \leq \frac{y(t)^2}{x(t)^2 + y(t)^2} = \sin^2(\theta(t)).$$

For $t \in \left[t_1, t_2\right[$, we have

$$
\begin{aligned}
\cot(\theta(t)) - \cot(\theta(t_1)) &= \int_{\theta(t)}^{\theta(t_1)} \frac{du}{\sin^2 u} = -\int_{t_1}^{t} \frac{\theta'(s)}{\sin^2(\theta(s))}\, ds \\
&\le \int_{t_1}^{t} 1\, ds = t - t_1 \le m\beta.
\end{aligned}
$$

Therefore, we get

$$
\cot(\theta(t)) - \frac{1}{(R^2-1)^{\frac{1}{2}}} \le \cot(\theta(t)) - y_1^{-1} \le m\beta, \quad \forall\, t \in [t_1, t_2[\,.
$$

This yields to a contradiction as $t \to t_2^-$ and therefore our claim is proved.

Adapting a result in [7, Lemma 2.2] to our situation, we have that for every $\varepsilon > 0$ there is $R^\varepsilon > 1$ such that

$$
|\mathrm{rot}_m(\zeta(0), 0) - \mathrm{rot}_m(\zeta(0), q_0)| < \varepsilon,
$$

holds for every point $q_0 = (a_{\bar n}, 0)$ with $||q_0|| \le 1$ and all solutions $\zeta(\cdot)$ with $||\zeta(t)|| \ge R^\varepsilon\,, \forall\, t \in [0, m\beta]$.

Thus there exists a sufficiently large radius $R^* > 1$ such that

$$
\mathrm{rot}_m(\zeta(0), q_0) < \frac{3}{4}, \quad \forall ||q_0|| \le 1, \forall \zeta \text{ with } ||\zeta(t)|| \ge R^*\,, \forall\, t \in [0, m\beta]
$$

and, by the result recalled at the beginning of the proof, we achieve the conclusion by taking

$$
R_0 \ge \eta(R^*).
$$

Thus the proof is complete. $\square$

As a next step, we evaluate the rotation numbers of small solutions around $q_0 = (a_{\bar n}, 0)$. To this end, we consider autonomous equation (10) that we write as a first order system

$$
\begin{cases}
\dot x = y \\
\dot y = gx - \bar n F_0(x).
\end{cases} \tag{15}
$$

System (15) is a conservative system with energy

$$
E_{\bar n}(x, y) = \frac{1}{2} y^2 - \frac{1}{2} g x^2 + \bar n \mathcal{F}_0(x), \quad \text{with } \mathcal{F}_0(x) = \int_0^x F_0(s)\, ds.
$$

For simplicity in the notation, in the sequel we set $E_{\bar n} = E$. Let us fix $d_0 > 0$ and $b \in]a, 1[$ such that

$$
F_0'(x) \ge d_0\,, \quad \forall\, x \in [a, b]
$$

and take

$$\mu_0 \geq \frac{g}{d_0}$$

such that

$$a < a_{\bar n} < b, \quad \text{for } \bar n > \mu_0 \,.$$

For $\bar n > \mu_0$ we have that $E'(\cdot,0)$ is strictly increasing on $[a,b]$ and vanishes in $a_{\bar n}$. Hence, on the interval $[a,b]$ we have that $E(\cdot,0)$ is a strictly convex function having absolute minimum at the point $a_{\bar n}$. If we take now a constant $c_{\bar n}$ with

$$E(a_{\bar n},0) < c_{\bar n} \leq \min\{E(a,0),E(b,0)\},$$

we have that the level line

$$\Gamma_{\bar n} = \{(x,y) : a \leq x \leq b,\ E(x,y) = c_{\bar n}\}$$

is a star-shaped curve around the equilibrium point q_0 . Moreover, for any point $q \in \Gamma_{\bar n}$ we have that the solution of system (15) with $(x(0),y(0)) = q$ is periodic and the corresponding orbit coincides with $\Gamma_{\bar n}$.

In order to avoid misunderstanding with the previously defined rotation number, we denote by $\mathscr{R}ot_m$ the rotation number associated to (10) along the time interval $[0,m\beta]$. If we denote by $\tau_{\bar n}$ the fundamental (i.e., minimal) period of $\Gamma_{\bar n}$, then we can conclude that

$$\mathscr{R}ot_m(q,q_0) \geq \left\lfloor \frac{m\beta}{\tau_{\bar n}} \right\rfloor, \quad \forall\, q \in \Gamma_{\bar n} \,.$$

We claim that $\lim\limits_{\bar n \to +\infty} \tau_{\bar n} = 0$.

Indeed, by the well known time-mapping formula we have

$$\tau_{\bar n} = \tau_{\bar n}^- + \tau_{\bar n}^+ = \sqrt{2}\int_{b_{\bar n}^-}^{a_{\bar n}} \frac{du}{(c_{\bar n} - E(u,0))^{\frac12}} + \sqrt{2}\int_{a_{\bar n}}^{b_{\bar n}^+} \frac{du}{(c_{\bar n} - E(u,0))^{\frac12}}$$

where

$$b_{\bar n}^- < a_{\bar n} < b_{\bar n}^+, \quad E(b_{\bar n}^-,0) = E(b_{\bar n}^+,0) = c_{\bar n} \,.$$

For $u \in [a_{\bar n}, b_{\bar n}^+]$, we have

$$\begin{aligned} c_{\bar n} - E(u,0) &= E(b_{\bar n}^+,0) - E(u,0) = \int_u^{b_{\bar n}^+} E'(x,0)\,dx \\ &= \int_u^{b_{\bar n}^+} (E'(x,0) - E'(a_{\bar n},0))\,dx = \int_u^{b_{\bar n}^+} \left(\int_{a_{\bar n}}^x E''(\xi,0)\,d\xi\right) dx \\ &\geq \frac{\bar n d_0 - g}{2}\,\left((b_{\bar n}^+ - a_{\bar n})^2 - (u - a_{\bar n})^2\right). \end{aligned}$$

Therefore, via an elementary integration, we obtain

$$\tau_{\bar{n}}^{+} \leq \frac{\pi}{\sqrt{\bar{n}d_0 - g}} \,.$$

A similar computation for $\tau_{\bar{n}}^{-}$ yields to

$$\tau_{\bar{n}} \leq \frac{2\pi}{\sqrt{\bar{n}d_0 - g}}$$

and this proves the claim. Hence we have that

$$\mathscr{R}ot_m(q, q_0) \geq \left\lfloor \frac{m\beta\sqrt{\bar{n}d_0 - g}}{2\pi} \right\rfloor, \quad \forall\, q \in \Gamma_{\bar{n}}\,,$$

which shows that $\mathscr{R}ot_m(q, q_0) \to +\infty$ with the order of $\sqrt{\bar{n}}$.

Now we are in position to complete the proof of Theorem 3.1.

Fix $m \geq 1$ and $N \geq 1$. Let $\mu = \mu_N \geq \mu_0$ be such that

$$\mathscr{R}ot_m(q, q_0) = \sigma_{\bar{n}} \geq \sigma > N, \quad \forall\, q \in \Gamma_{\bar{n}}\,,$$

holds for each $\bar{n} > \mu$. Fix also an interval $[\mu_1, \mu_2] \subset]\mu, +\infty)$ and consider equation (9) with $\bar{n} \in [\mu_1, \mu_2]$. By the continuous dependence of the solutions from the coefficients and from initial data (otherwise, some direct estimates may also be performed) we have that there exists $\varepsilon_1 > 0$ (depending on μ_1 and μ_2 and, in turns, also on N) such that for each forcing term p with $|p|_1 < \varepsilon_1$, condition (12) is satisfied as well as $\mathrm{rot}_m(z_0, q_0) > N$ holds for each $q_0 = (a_{\bar{n}}, 0)$ and $z_0 \in \Gamma_{\bar{n}}$ provided that $\bar{n} \in [\mu_1, \mu_2]$. Recalling the definition of c_0 as a bound for $|F_0|$, we have that for

$$|\tilde{n}|_1 < \varepsilon = \frac{\varepsilon_1}{c_0}\,,$$

it follows that (12) holds and

$$\mathrm{rot}_m(z_1, q_0) > N, \quad \forall\, z_1 \in \Gamma_{\bar{n}}.$$

For any chosen $n(x)$ with $\bar{n} \in [\mu_1, \mu_2]$ and $|\tilde{n}|_1 < \varepsilon$, we can take a sufficiently large radius R_n such that $\Gamma_{\bar{n}} \subseteq B(q_0, R_n)$ and

$$\mathrm{rot}_m(z_2, q_0) < 1, \quad \forall\, ||z_2|| = R_n\,.$$

The Poincaré - Birkhoff theorem (Theorem 3.2) ensures the existence of $2N$ solutions which are $m\beta$-periodic. Clearly, for $m = 1$ we have exactly the result claimed in Theorem 3.1 and thus the proof is complete. □

Clearly, from the above proof we have a result about the existence of subharmonic solutions (of period $m\beta$ with $\beta \geq 2$) as well. Indeed, the discussion about the minimality of the period given in Remark 3.1 yields to:

Theorem 3.3 *Assume that*

$$F'(a) > 0.$$

Then, for every integer $m \geq 2$ and each $K \geq 1$, there is a (large) value $\mu = \mu_K > 0$ such that for each $\mu_2 \geq \mu_1 > \mu$ there is a (small) value $\varepsilon = \varepsilon_{K,\mu_1,\mu_2} > 0$ such that for each $n(\cdot)$ with

$$\bar{n} \in [\mu_1, \mu_2] \quad \textit{and} \quad |\tilde{n}|_1 < \varepsilon$$

there are at least $2K$ solutions of equation (1) which are $m\beta$-periodic and take values in $]0,1[$. Moreover, such solutions do not belong to the same periodicity class and also they are not $i\beta$-periodic, for each $i = 1, 2, \ldots, m-1$, so that the period $m\beta$ is minimal.

Proof. Once we have fixed m and K, we take the set

$$\mathcal{N} := \{l_1 = 1 < l_2 < \cdots < l_K := N\}$$

made by the first K numbers which are co-prime with m.[1] According to the conclusion of the proof of Theorem 3.1, we find that there are $2N \geq 2K$ periodic solutions of period $m\beta$. As a consequence of Remark 3.1 we know that these $2N$ solutions appears in pairs which are characterized by the fact that they have, respectively, $2, 4, 6, \ldots, 2N$ oscillations in the time interval $[0, m\beta[$. Such oscillations are associated to the rotation numbers of the solutions (an information which comes from the use of the Poincaré - Birkhoff fixed point theorem). As we already explained in Remark 3.1 we are sure that for those solutions having a rotation number ℓ which is co-prime with m, it holds that $m\beta$ is the minimal period. By the choice of N and the set $\mathcal{N}$ we are lead to conclude that at least $2K$ among the $2N$ periodic solutions (that is those with rotation numbers equal to $l_1, l_2, \ldots, l_K$) are of minimal period. This observation concludes our proof. $\square$

We end our paper, by showing that it is possible to realize a "good" splitting of the weight function $n(x)$ for a profile like the one considered in [3].

Example 3.1 Consider now equation (1) and assume, like in [3] that $n(x)$ is a periodic piecewise constant function satisfying

$$n(x) = \begin{cases} n_1 & \text{if } x \in]0, \alpha[\mod \beta \\ n_0 & \text{if } x \in]\alpha, \beta[\mod \beta \end{cases} \tag{16}$$

with $0 < n_0 < n_1$. Like in [3] we assume that F is a smooth function satisfying the sign conditions we have already described at the beginning, that is $F(0) = F(a) = F(1) = 0$ with $a : 0 < a < 1$ and such that $F(s) > 0$ for

[1] Two positive integers l and m are co-prime (or relatively prime) if $\mathrm{GCD}(l, m) = 1$

$s \in (-\infty, 0[\cup]a, 1[$, $F(s) < 0$ for $s \in]0, a[\cup]1, +\infty)$. In order to apply our result, we also suppose that $F'(a) > 0$ (an hypothesis which is always satisfied by the nerve fiber models considered in [3], [10]). In [16] we have already proved the existence of threshold values, associated to a general shape of $n(x)$ which imply, for the particular case of (16), that the only periodic solution is the trivial one when n_0 and α are sufficiently small, while we proved the existence of at least two nontrivial β-periodic solutions (under a weak technical assumption on $F(s)$) when n_1 is sufficiently large and α is close to β. Now, in view of Theorem 3.1 and Theorem 3.3 we propose the splitting

$$n(x) = n_1 + \tilde{n}(x)$$

where

$$\tilde{n}(x) = \begin{cases} 0 \quad \text{if } x \in]0, \alpha[\quad \bmod \beta \\ n_0 - n_1 \quad \text{if } x \in]\alpha, \beta[\quad \bmod \beta \end{cases}$$

By this choice $\bar{n} = n_1$ and, moreover, we have

$$|\tilde{n}|_1 = |\tilde{n}|_{L^1([0,m\beta])} = m(n_1 - n_0)(\beta - \alpha).$$

In spite of the fact that we take $\bar{n} = n_1$ large, we are allowed to make $|\tilde{n}|_1$ as small as we like, by taking α sufficiently close to β. Hence Theorem 3.1 and Theorem 3.3 can be applied and the existence of a large number of harmonic and "true" subharmonic solutions for the equation (1) is guaranteed.

References

[1] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14** (1973), 349–381.

[2] P. Buttazzoni and A. Fonda, Periodic perturbations of scalar second order differential equations, *Discrete and Continuous Dynamical Systems* **3** (1997), 451–455.

[3] P.-L. Chen and J. Bell, Spine-density dependence of the qualitative behavior of a model of a nerve fiber with excitable spines, *J. Math. Anal. Appl.* **187** (1994), 384–410.

[4] C. De Coster, *The Method of Lower and Upper Solutions for ODEs*, SISSA, Academic Year 1994–1995.

[5] C. De Coster and P. Habets, Upper and lower solutions in the theory of ODE boundary value problems: Classical and recent results. *In: Non linear analysis and boundary value problems for ordinary differential equations*, CISM Courses and Lectures, Vol. **371**, Springer Wien, New York (1996), 1–78.

[6] T. Ding, An infinite class of periodic solutions of periodically perturbed Duffing equations at resonance, *Proc. Amer. Math. Soc.* **86** (1982), 47–54.

[7] T. Ding and F. Zanolin, Subharmonic solutions of second order nonlinear equations: a time-map approach, *Nonlinear Analysis* **20** (1993), 509–532.

[8] W. Ding, A generalization of the Poincaré - Birkhoff theorem, *Proc. Amer. Math. Soc.* **88** (1983), 341–346.

[9] A. Fonda and F. Zanolin, Periodic oscillations of forced pendulums with very small length, *Proc. Roy. Soc. Edinburgh Sect. A* **127** (1997), 67–76.

[10] P. Grindrod and B.D. Sleeman, A model of a myelinated nerve axon: threshold behaviour and propagation, *J. Math. Biology* **23** (1985), 119–135.

[11] A.R. Hausrath and R.F. Manásevich, Periodic solutions of a periodically perturbed Lotka-Volterra equation using the Poincaré-Birkhoff theorem, *J. Math. Anal. Appl.* **157** (1991), 1–9.

[12] M.A. Krasnosel'skiĭ, *The Operator of Translation along the Trajectories of Differential Equations*, Amer. Math. Soc., Providence, R.I., 1968.

[13] C. Rebelo and F. Zanolin, Multiplicity results for periodic solutions of second order ODEs with asymmetric nonlinearities, *Trans. Amer. Math. Soc.* **348** (1996), 2349–2389.

[14] W. Neumann, Generalizations of the Poincaré - Birkhoff fixed point theorem, *Bull. Austral. Math. Soc.* **17** (1977), 375–389.

[15] C. Zanini, Rotation numbers, eigenvalues, and the Poincaré - Birkhoff theorem, *J. Math. Anal. Appl.* **279** (2003), 290–307.

[16] C. Zanini and F. Zanolin, Positive periodic solutions for ordinary differential equations arising in the study of nerve fiber models. *In: Applied and Industrial Mathematics in Italy*, Proceedings of the 7th SIMAI Conference, Venice 2004 (M. Primicerio, R. Spigler and V. Valente, eds.), Series on Advances in Mathematics for Applied Sciences, Vol. **69**, World Scientific, Singapore (2005), 564–575.

[17] F. Zanolin, Continuation theorems for the periodic problem via the translation operator, *Rend. Sem. Mat. Univ. Pol. Torino* **54** (1996), 1–23.

[18] F. Zanolin, Time-maps and boundary value problems for ordinary differential equations. *In: Tricomi's ideas and contemporary applied mathematics* (Rome/Turin, 1997), Atti Convegni Lincei, Vol. **147**, Accad. Naz. Lincei, Rome (1998), 281–302.